Dynamical Systems Problems Set - Problem set # 1

REU 2021

1. The basic fact of life is that more specimens of the same species there are, the more they reproduce. So naively, one would imagine that the basic model for growth is that the size of the population $N(t)$ increases linearly at a rate $r > 0$, i.e.,

$$\dot{N}(t) = rN \,, \qquad N(0) = N_0 \geq 0 \,. \tag{1}$$

   (a) Find the solutions of the naive, linear equations. Would there be growth if there are no specimens at $t = 0$?

   (b) But this is a simplistic model. Suppose we have a meadow and a population of gazelles. At some points, if there are too many gazelles, food will be scarce and the gazelles would start to perish. So the rate of growth *decreases* with N. To model that, take $r \mapsto r(1 - N/k)$, which is **The logistic model**

$$\dot{N} = rN(1 - N/k) \,, \qquad N(0) = N_0 \geq 0 \,. \tag{2}$$

   Plot the **Phase portrait** to this equation, classify the fixed points and different regimes, and ask yourself - do these regimes make sense from a modeling point of view?

   (c) A team of exhausted epidemiologists sat down at 1am to write down a model for population growth. Instead of the usual logistic equation, they wrote down something quite different. Find a solution for each of the following equations due to which it cannot be a model the population size $N(t)$ (parameters are always assumed to be positive).

      i. $\dot{N}(t) = rN - k$

      ii. $\dot{N} = N + r$

      iii. $\dot{N} = N^{1/5}$

   (d) Even though you already classified the fixed points of the original logistic model, corroborate your observations using linear stability analysis.

2. Life is not just saddle point bifurcations! In this example we will see a **transcritical bifurcation**. Consider

$$\dot{x}(t) = rx - x^2 \,.$$

   Separate your analysis to different values of r and plot the bifurcation diagram.

3. We will now venture into 2d systems. Consider the system

$$\dot{x}(t) = -y + ax(x^2 + y^2) \,,$$
$$\dot{y}(t) = x + ay(x^2 + y^2) \,.$$

Question: Is $(0, 0)$ a stable fixed point?

To answer this question, first write the ODE in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$. Use the fact that $x^2 + y^2 = r^2$ to arrive to a decoupled set of ODEs for r and θ. Find the stability of the origin as a function of a.

4. While we are on the subject of 2d ODEs, consider the following system in polar coordinates:

$$\dot{r} = r^2 - r^4) \,,$$
$$\dot{\theta} = 1 \,.$$

   (a) Consider the first equation as a scalar equation. Plot its phase diagram and classify the fixed points:

   (b) Now recall that these are in fact polar coordinates. How do the different fixed points in r manifest themselves in the dynamics of the system as $t \to \infty$?