DYNAMICAL SYSTEMS MINI-COURSE FOR COLUMBIA'S MODELLING REU

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1. Dynamical systems - what and why

How do quantities change in time? This is an overarching fundamental question in the sciences, from physics, to biology, the life sciences, computation, finance, and the social sciences. How fast does an electric capacitor discharge? Can the motocyclist jump over the river? When will a pandemic stop its spread? Will the S&P 500 make profit over the next year? over the next hour? All of these questions have to do with *dynamical systems.*

We will mostly concentrate on a type of dynamical systems where time *changes continuously.* These are best exemplified by Newton's second law of motion $\ddot{x}(t) = \dot{v} = F(t, x, \dot{x})/m$. The position changes according to the velocity, and the velocity changes according to the force applied to it. Force changes as a function of time, position, and velocity. This is an **Ordinary differential equation** (ODE). It prescribes the momentarily rate of change. Its solution is a function in time, rather than a number.

In any first course on ODEs, one learns how to solve some special types of ODEs. Take a simple example: $\dot{x}(t) = ax$ where $x(0) = x_0 \in \mathbb{R}$. We know what the solution is, $x(t) = x_0 e^{at}$. But as it turns out, **there are very few equations we can solve in closed form**. It takes almost a miracle to achieve that. To see why this is true, think of a simple calculus task you know - integration. Even if f is as smooth as we would like, we can rarely compute $\int_a^b f(x)\,dx$ in closed form, and we rely almost entirely on guesses, tricks and experience. If you believe that integration is hard, then take a separable, scalar, ODE

$$\frac{dy}{dt} = f(t)g(y)\,, \qquad y(0) = y_0 \in \mathbb{R}\,.$$

The solution satisfies, on a very formal level

$$\int\limits_{y(0)}^{y(t)} \frac{1}{g(y)}\,dy = \int\limits_{0}^{t} f(\tau)\,d\tau\,.$$

In a primer on ODEs, f and g are cooked up precisely so that these two integrals could be computed. But in applications, it is likely that neither will have a closed form expression.

One way to solve these equations is numerically, using a computer. The ability to solve ODEs (and PDEs) numerically revolutionized the sciences, by allowing us insight into previously intractable problems. While this is a tool that we will learn about, it is not an unlimited one. Simulations suffer from inaccuracies (we get something on the computer, but does it relate to the ODE?), inefficiencies (some simulations just take too long to run), and only allows to experiment with *some* parameters.

So closed form solution of ODEs are hard to come by, and numerical solutions of ODEs are both difficult and limited. There is a small opening through which we can enter; we can analyze ODEs without fully solving them. For example, we can ask about the limit of the dynamics as $t \to \infty$. We can ask geometrical questions about the trajectories - are they bounded? Are they almost periodic? Are they "all over the place" (chaotic, ergodic, mixing, etc.)? As it turns out, one can answer these questions about the *global in time* behavior of $x(t)$ just by examining the *local in time* rules that govern them, the ODEs.

2. Linear systems

Going back to $\dot{x}(t) = ax$, we already see some interesting properties of the solution $x(t) = x_0 e^{at}$. What is the limit of x as $t \to +\infty$? It is x_0 is $a = 0$ or $x = 0$, it is 0 if $a < 0$, and $\pm\infty$ if $a > 0$, depending on the sign of x_0. On the one hand, the long but finite time dependence on a is smooth, but on the other hand, the infinite-time limit is not!

The next natural extension of a one-dimensional time-independent/autonomous linear system is an n-dimensional autonomous linear system, written succinctly in matrix form

$$\dot{\underline{x}}(t) = A\underline{x}\,, \qquad \underline{x}(0) = \underline{x}_0 \in \mathbb{R}^n\,, \qquad A \in M_n(\mathbb{R})\,.$$

As you well know, solutions of this initial value problem (IVP) can be written as

$$\underline{x}(t) = e^{At}\underline{x}_0\,,$$

where the matrix exponent can be defined using Taylor's series

$$e^A \equiv \sum_{j=0}^{\infty} \frac{A^j}{j!}\,.$$

But this form does not reveal much about the dynamics. To make this expression more insightful, suppose that A is unitarilly diagonalizable, i.e.,

$$A = U^* D U \,, \qquad \det(U) = \pm 1 \,, \qquad D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \,.$$

Then

$$\underline{x}(t) = U^* e^{Dt} U \underline{x}_0 \,.$$

This makes an intuitive sense: because A is diagonlizable, even though A might seem complicated, there exists an orthonormal system of coordinates $\underline{y}_1, \ldots, \underline{y}_n \in \mathbb{R}^n$ such that if we represent $\underline{x}(t) = \sum c_j(t) \underline{y}_j(t)$, then the system of ODEs simplifies to

$$\dot{c}_j(t) = \lambda_j c_j \,, \qquad c_j(0) = \langle \underline{y}_j, \underline{x}_0 \rangle \,, \qquad 1 \le j \le n \,.$$

These are simple 1d ODEs that we can solve, and game over. **Comments:**

(1) It is an unfortunate fact that real matrices are not necessiraly diagonalizable, and certainly not over the $\mathbb{R}$. Hence, D might be complex valued. This is a source of richness - consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Longrightarrow \lambda_1 = i \,, \qquad \lambda_2 = -i \,,$$

for which we get periodic solutions of the form $e^{\pm it}$, or really (for real initial conditions) $\cos(t)$ and $\sin(t)$.

(2) Diagonalizing matrices can be hard. In general, finding the eigenvalues of A involves finding roots of the n-th degree polynomial $0 = \det(A - xI_n)$.

(3) **Partial Differential Equations (PDEs)**. While we will not attempt to cover this entire topic, linear ODEs do give us some insight into the world of PDEs. Consider the following example, the **heat equation**:

$$\partial_t u(t,x) = \partial_{xx} u \,, \qquad u(0,x) = f(x) \,,$$

with a periodic boundary condition

$$u(t,x) = u(t,x+1) \,, \qquad \forall x \in [0,1) \,, \quad t \in \mathbb{R} \,.$$

This is a PDE and not an ODE, since it involves partial derivatives (in time and in space). Still, humor me: since u is periodic in space, we know from Fourier analysis that we can express it by a Fourier series. If you have never seen this, just wait: under suitable regularity conditions, a periodic function can be written as the sum of elementary periodic functions, in this case cosines and sines, i.e.,

$$f(x+1) = f(x) \Longrightarrow f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}\,, \qquad \hat{f}(n) \in \mathbb{C}\,.$$

That the right hand side converges, how to find the coefficients $\hat{f}(n)$, and other interesting questions, are left to another time.

Now a true miracle occurs: consider each $e_n(x) \equiv e^{2\pi i n x}$ as a basis function. Note that

$$\partial_{xx} e_n(x) = (2\pi i n)^2 e_n(x) = -4\pi^2 n^2 e_n(x)\,.$$

Moreover, on a suitable space, ∂_{xx} is a linear operator ($\partial_{xx}(f + 2g) = f'' + 2g''$). So we now know that the basis $\{e_n(x)\}_{n\in\mathbb{Z}}$ **diagonalizes the right hand side of the heat equation**, i.e., it decouples under this choice of basis (compare to the $\underline{y}_j$'s from the matrix case). Since solutions remain periodic for all times, we can write

$$u(t,x) = \sum_{n=-\infty}^{+\infty} \hat{u}(t,n) e_n(x)\,, \qquad \hat{u}(0,n) = \hat{f}(n)\,,$$

and find that for all n

$$\hat{u}_t(t,n) = \hat{u}(t,n)\partial_x x e_n(x) = \hat{u}(t,n)(-4\pi^2 n^2) e_n(x)\,, \qquad \hat{u}(0,n) = \hat{f}(n)\,.$$

This is an **infinite system of 1d linear ODEs**, which we can solve, yielding

$$u(t,x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x} e^{-4\pi^2 n^2 t}\,.$$

This was a long exercise. What did we learn from it? That linear ODEs are the basis for linear PDEs. We "only" need to understand the eigenvalues of the time-independent operator, and then solve simple ODEs. Why is this not the end of the course in PDEs? First, we really haven't proved anything, but also - how does one find these eigenvalues? Why should they always exist?

3. Nonlinear ODEs

"Using a term like nonlinear science is like referring to the bulk of zoology as the study of non-elephant animals." (S. Ulam)

It is really the exception, rather than the rule, that a system of ODEs is linear. The principle behind

$$\dot{x} = f(x)\,,$$

is "the rate of change of a state x depends on the state x via the map f". Why then should f be linear?

In this minicourse we will mostly consider autonomous equations, i.e., where $f = f(x)$ and not $f(x,t)$. Modeling-wise, this means that the system is independent of any external "clock". Autonomous systems are interesting too (they are a focus of my research!) but beyond our scope.

The first new tool that we want to introduce is the geometric way of thinking and **phase-diagrams**. Consider the **initial value problem (IVP)**[1]

$$\dot{x} = \sin x\,, \qquad x(0) = x_0 \in \mathbb{R}\,. \tag{1}$$

One can solve it by separability and get an implicit solution:

$$\begin{aligned} dt &= \frac{dx}{\sin x} \\ t - 0 &= -\log|\csc x(\tau) + \cot x(\tau)||_{\tau=0}^{\tau=t} \\ t &= \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x(t) + \cot x(t)}\right|\,. \end{aligned}$$

This is an exact, but implicit, solution. Fantastic. But what do we learn from it? If, for example, we want to know how solutions behave as $t \to +\infty$, we learn nothing!

3.1. **Graphical approach to fixed points.** First note something nice about the ODE. If for some reason $x_0 = k\pi$ for some integer $k \in \mathbb{Z}$, then $\dot{x}(0) = \sin(k\pi) = 0$. So the equation does not change. If it does not change, $x(t)$ remains constant, and $\dot{x}(t) = 0$ for all $t > 0$. This line of argument might seem suspicious at first - couldn't $\dot{x}(t)$ "jump"? As applied-minded people, we believe that if $x_0 = 0$, and $\dot{x}(0) = 0$, then $x(t) \equiv x_0$ for all times.

The *rigor* of the argument comes from a key concept, which is **existence and uniqueness.** We will quote, but not prove, a theorem by Picard:

Theorem 1 (Picard-Lindelöf Uniqueness Theorem)**.** *Consider the ODE*

$$\dot{\underline{x}}(t) = f(t, \underline{x})\,, \qquad \underline{x}(t_0) = \underline{x}_0 \in \mathbb{R}^n\,. \tag{2}$$

[1] We will use here the following convention: ODEs are the local dependence $\dot{x} = f(x)$. IVPs are ODEs + initial conditions. This is not the only possible convention, but it is a useful one.

Let D be an open box in $\mathbb{R} \times \mathbb{R}^n$ such that $(t_0, x_0) \in D$. Then if f is continuous in D **and Lipschitz continuous in** *x, then there exists $\tau > 0$ and* **a unique** *$x(t) : (t_0 - \tau, t_0 + \tau) \to \mathbb{R}$ that satisfies* (2)*.*

The proof is beyond the scope of this mini-course, but consider the implication of this theorem to (1). We identified that if $x_0 = 0$, then the solution "stays" there. Now we note that $x(t) = 0$ solves (1) for all times t. Now, suppose that another solution starts at $x_0 = 0$. Then locally there is a *unique* solution around $t = 0$, which is $x(t) = 0$. But this argument can be extended to all times, since $f(x) = \sin(x)$ is smooth for all t and x.

What about other values of x_0? Suppose $0 < x_0 < \pi$. Then $\dot{x}(0) = \sin x_0 > 0$, and therefore the $x(\Delta t) > x_0$ for small $\Delta t > 0$. We see that the derivative is always positive, so $x(t)$ is pushed towards π. But note that as $x(t) \to \pi$, $\dot{x} \to 0^+$, and therefore we now see that $\lim_{t\to\infty} x(t) = \pi$. By plotting $\dot{x}$ as a function of x, we see that when k is odd, $k\pi$ are **attractive/stable fixed points**, i.e., a small deviation in x_0 from $k\pi$ them prescribes a trajectory which converges to $k\pi$. The same cannot be said of $k\pi$ for *even* k. Then, these points are **repelling/unstable fixed points**.

The key concept learned by this example is that of **phase space and phase portrait**. In an ODE of the form $\dot{x} = f(x)$, the derivative only depends on x. Therefore, by plotting $f(x) = \dot{x}$ as a function of x, we can understand the possible *asymptotics* and fixed points structure of the solutions $x(t)$ based only on x_0. **Question:** Does it matter that x_0 is given in $t = 0$ and not in $t = 532/e$?

Another question: Can we learn the $t \to -\infty$ dynamics from the phase portrait?

4. Linear stability analysis

The question we will deal with now is whether we can say that a fixed point is stable or unstable *without* the phase portrait. The algebraic method will complement the graphic one in cases where the algebra is simple and the sketching is hard. What we do here really only sets the stage to the world of linear stability analysis, which goes through to higher-order ODEs and PDEs, and remains an active field of study to this day.

Suppose x^* is a fixed point of $\dot{x}(t) = f(x)$. Define $\eta(t) = x(t) - x^*$. Does η shrink or blow-up for $\eta(0)$ small?

By differentiation $\dot{\eta}(t) = \dot{x}(t) - \dot{x}^* = f(x) - 0 = f(x^* + \eta)$. If f is C^1, then by Taylor expansion:

$$\dot{\eta} = f(x^*) + \eta f'(x^*) + O(\eta^2) = \eta f'(x^*) + O(\eta^2)\,.$$

So, if we start with $|\eta(0)| \ll 1$, then $\dot{\eta}(t) \approx f'(x^*)\eta$, which is a linear ODE. Therefore $\eta(t) \approx \eta(0)e^{f'(x^*)t}$, and so if $f'(x^*) < 0$ then η contracts and vanishes with time, and if $f(x^*) > 0$ it blows up with time. These two cases correspond to a stable and unstable fixed points, respectively.

Rigor alerts:

(1) When $f'(x^*) < 0$, then $\eta \to 0$ and therefore the error term $O(\eta^2)$ just becomes increasingly negligible. If $f'(x^*) > 0$, the error term becomes *less* negligible, and therefore the linearization becomes invalid after some time. Still, it does not matter, since to be an unstable fixed point, it is sufficient to see that small difference function η increase with time and does not decrease.

(2) We used a Theorem that we did not prove - if $|\dot{\eta}(t) - f(x^*)| < c\eta^2$, and $\eta(0)$ is sufficiently small, then one can get an inequality on the solution. Such *differential inequalities* need to be proven.

5. Bifurcations

Bifurcation, in ODEs, means the study of **parametric families of ODEs**, and how the dynamics change qualitatively with the parameter. The ODEs $\dot{x}(t) = x$ and $\dot{x}(t) = 1.001x$ differ, and certainly as $t \to \infty$ their solutions diverge exponentially, but this is not a very interesting difference. The dynamics are the same.

A trivial example would be $\varepsilon\dot{x}(t) = 0$ with $x_0 = 1$, which has a single solution for $\varepsilon > 0$, but has infinitely many solutions for $\varepsilon = 0$. But somehow that seems to miss the point of ODEs and models.

What are non-trivial but interesting examples?

5.1. **Saddle point bifurcation.**

Consider the ODE $\dot{x}(t) = r + x^2$. By the phase diagram, we see three very different regimes:

(1) $r < 0$, the equation has two fixed points $\pm\sqrt{r}$, the negative one stable and the positive one unstable.

(2) $r > 0$, the equation $x^2 + r = 0$ has no real solutions and so all solutions tend to ∞ as $t \to \infty$.

(3) $r = 0$, the solution has a single half-stable fixed point $x^* = 0$: when $x_0 > 0$ the solution blows up, and when $x_0 < 0$ it converges to x^*.

Here comes a graphical explanation on how to plot Bifurcation diagrams and the resemblance of the diagram to a saddle.

The idea is that if we think on how fixed points “move” with the parameter (which can be many things - mass, spring constant, etc.), we have to allow for bifurcations - points disappear, appear, merge and so on.

A moment of intuitive and imprecise generalizations: bad things in math happen on three occasions: when we try and take even roots of negative numbers, when we divide by zero, and when we sum a diverging series. Look out for these and you will find many many bifurcations.

5.2. **Normal forms** **(if time allows).** Now we are going to see some really wacky stuff. Suppose we have $\dot{x} = r + x^2$ and $\dot{x} = -3.5r + x^2$. Those are different ODEs, but the first with $r = -3.5$ is identical to the latter with $r = 1$. In math and in science, we look for canonical forms, and if not for canonical forms - for equivalence classes and universality. In fact, we expect that even for a very different ODE, $\dot{x} = r - x - e^{-x}$, since it has a saddle point bifurcation (see HW), we will be able to rescale it and change variables so it will look like the former two.

We expect points to appear/disappear for equations like $\dot{x}(t) = f(x; r)$ when $f(\cdot; r)$ intersects, or does not intersect, the x axis, since a generic minima looks like x^2 (Taylor). More formally:

Let $\dot{x} = f(x)$ and let r_c be the parameter an where a single half-stable fixed point appears at x^*, and around which we have a saddle point bifurcation. Then for some x, r near r_c and x^*, we have

$$
\begin{aligned}
\dot{x}(t) &= f(x; r) \\
&= f(x*; r) + (x - x^*)\partial_x f(x^*, r_c) + (r - r_c)\partial_r f(x^*; r_c) + \frac{1}{2}(x - x^*)^2 \partial^2_{x^2} f(x^*; r_c) + O(|x - x^*|^3, |r - r_c|^2)\,.
\end{aligned}
$$

Now - $f(x^*; r_c) = 0$ since its a fixed point. $\partial_x f(x^*; r_c) = 0$ also since $f(x; r_c)$ has to be strictly positive (or negative) on both sides of $f(x^*; r_c)$, ans so x^* is a local minimum (or a maximum) for $f(\cdot; r_c)$. We are left with the leading order expressions which can be written as

$$
\dot{x} = a(r - r_c) + b(x - x^*)^2 + \text{h.o.t.}\,.
$$

So locally around the bifurcation, everything looks like the “original” bifurcation we observed. This is incredible!

6. Numerical schemes for ODEs

This is a whole world of research, spanning from the invention of modern computers to present days. Today, we will survey the simplest possible method. This method hints at more realistic methods in use. When in need, you can always try ready-made methods by Matlab (‘ode45’) or SciPy (‘ scipy.integrate.RK45’), but it is important to at least to understand the principles:

6.1. **Forward Euler.** Consider the scalar ODE

$$\dot{x}(t) = f(x,t)\,,$$
$$x(t_0) = x_0\,.$$

One can write the solution in integral form

$$x(t) = x_0 + \int_{t_0}^{t} f(x(t'),t')\,dt'\,.$$

It may seem like we solved the ODE, but in reality we are integrating over $x(t')$, which we do not know. Suppose $t - t_0 = \Delta t$ is very small. Then we can approximate the integral by the point we already know; call the approximate solution $y(t)$, then

$$x(t_0 + \Delta t) \approx x_0 + y(t_0 + \Delta t) \equiv \int_{t_0}^{t_0+\Delta t} f(x(t_0),t_0)\,dt' = x_0 + f(x_0,t_0)\Delta t\,.$$

We can then proceed in a piecewise fashion, and write for every $t_j = t_0 + j\Delta t$ that

$$y_{j+1} = y(t_{j+1}) = y_j + f(y_j,t_j)\Delta t\,.$$

This is it, this is the algorithm. There are (at least) two questions we can ask:

(1) **Accuracy:** As $\Delta t \to 0$, does $y(t_j) \to x(t_j)$. More quantitatively, how big is the error? Usually, the answer is in terms of norms of f times $(\Delta t)^r$, where the exponent r changes according to the algorithm.
(2) **Stability.** Fix Δt, and suppose we have a slight error in the initial condition (due to, e.g., rounding error) $y(t_0) = x_0 + \varepsilon$. Does $y(t)$ deviates from $x(t)$ as $t \to \infty$?

In general, Forward Euler is not the top of its class in either criterion. Other schemes, like Runge-Kutta and Crank-Nicolson are variation on the same themes which achieve better accuracy and better stability.

6.2. **Bifurcation - a numerical study.** Suppose we want to plot a bifurcation diagram based on numerical experiment. How should we go about it? At our disposal is a "machine" (piece of code) $F(r,x_0)$ that solves and ODE based with parameter(s) r and initial condition(s) x_0. To even begin, we have to have an educated guess (e.g., based on the equation) as to what range of parameters r are we looking at. We will propose a method in a top-down fashion:

(1) Every point in the bifurcation diagram is on the plane of $x - r$, and corresponds to a solution of an ODE. But we cannot solve everywhere.

(2) Fix r in the range. There are two ways to find fixed points:

   (a) Trying many initial conditions, run the ODE (numerically) for each one, and see where it converges to. Downside- finds only stable fixed points.

   (b) Use some root-finding algorithm (e.g., Newton-Raphson) to find the roots of $F(r, \cdot)$. Then, asses stability by perturbing around the roots and simulate the ODE for a sufficiently long time.

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